Overview of Inverse Problems

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Parameter Identification in Civil Engineering

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Abstract

This booklet relates the major developments of the evolution of inverse problems. Three sections are contained within. The first offers definitions and the fundamental characteristics of an inverse problem, with a brief history of the birth of inverse problem theory in geophysics given at the beginning. Then, the most well known Internet sites and scientific reviews dedicated to inverse problems are presented, followed by a description of research undertaken along these lines at IFSTTAR (formerly LCPC) since the 1990s. The final section concerns the different approaches available to solve an inverse problem. These approaches are divided into three categories: functional analysis, regularization techniques for ill-posed problems, and stochastic or Bayesian inversion.

Keywords: inverse problems, parameters identification, global optimization, modal identification, wavelets, tomography, mechanical engineering, civil engineering

Introduction - Definitions

The term '**inverse problem**' appeared in the 1960s, notably to designate in geophysics the determination, through input-output or cause-effect experiments, of unknowns in the physics equations. Today the notion of 'inverse problem' designates the best possible reconstruction of missing information, in order to estimate either the loads (**identification of sources or of the cause**), or the value of undetermined parameters (**identification of model parameters**).

Le Verrier's discovery of the planet Neptune is a famous example of an inverse problem that demanded an identification of the cause. At the beginning of the 19th century, astronomers had not yet discovered the planet Neptune. The most distant planet then discovered was Uranus, found by Herschel in 1781. When astronomers applied Newton's theory of universal gravitation to the movement of Uranus, the calculated orbit and movement, when taking only into account the disturbances caused by Jupiter and Saturn, did not match the observed orbit and movement. Encouraged by Arago, Le Verrier began in 1844 to labor away at this calculation, using inverse perturbations of features (mass, orbit, current position) of the hypothetical planet assumed to produce Uranus' observed irregularities. In his report to the Academy of Sciences on August 31, 1846, Le Verrier presented the orbital elements of this new planet. Then, on September 23, the exact day when he received from Le Verrier a letter detailing its predicted position, the Berlin astronomer Galle succeeded in observing the object. Arago then declared to the Academy of Sciences: "Mr. Le Verrier perceived the new star without even throwing a single glance toward the sky; he saw it at the end of his quill."

Before moving on to the mathematical specificities of inverse problems, let us examine some key related terms. The situation in which the parameters or sources are directly accessible by measurement is called **direct measurement**. However, in many cases, the information searched for is not directly measurable. Instead, it is physically connected to other measurable values. We call this situation **indirect measurement**. Though these measurable values are evidently dependent on the physics of the studied phenomenon, they are also dependent on the measurement instruments employed. The laws of physics that link observable information to the values searched for are generally mathematically complex, using, for example, integral equations or partial derivatives. The solution to a problem that calculates observable effects from unknown values, or a **direct problem**, is often simpler and more easily mastered than that of an **inverse problem**, which calculates unknown values from observable effects.

To start out, we can say that an inverse problem arises from the confrontation of experimental data with results from numerical simulations. The problem consists in seeking to minimize the distance between the measurement and the calculation.
One characteristic of inverse problems is being **ill posed** or **incorrectly posed**, in the sense of Hadamard: the total measured data does not allow the existence of a solution to the problem, the solution is not unique, or, even further, due to disturbance in the data the solution is not stable.
If the data from measurements can in theory create a space of either finite or infinite dimensions, in practice the data are always finite and discrete. When the number of parameters in a model is smaller than the number of data points from the measurements, the problem is called **overdetermined**. In this case, it may be possible to add a criterion that diminishes or eliminates the effects of aberrant data. On the other hand, if the problem consists in determining continuous parameters that are thus sampled from a very large number of values, and if the number of results from the experiments is insufficient, the problem is called **underdetermined**. It is then necessary to use a priori information to achieve a reduced number of possible solutions, or, in the best case, only one. Since for an underdetermined problem there are often several possible solutions, it is necessary to specify the confidence level that one can give to each solution. For these problems, the data can also be affected by a likelihood coefficient or probabilistically weighted. If this is the case, a **Bayesian approach** can be used for the problem.

Finally, another definition for an inverse problem could be: an ill-posed problem that has for its objective the inversion of a physical model by means of a partial image of that model's effects.

Areas of Use – Historical Development

Inverse problems form an essentially multidisciplinary scientific field, combining mathematics with multiple branches of physics. The pioneering works in the 1970s on the solving of ill-posed problems (notably those of Russians like A.N. Tikhonov [TA77], and more recently his students [BM91]), and next the analysis and execution of inversion strategies for ill-posed problems, helped create a now widely recognized branch of applied mathematics. One finds numerous applications of inverse analysis in the physical and mechanical sciences: a non-exhaustive list includes optics, radar, calorimetry, spectroscopy, geophysics, meteorology, oceanography, acoustics, radioastronomy, non-destructive control, biomedical engineering, instrumentation, imaging, civil engineering and mechanical engineering. Work undertaken at IFSTTAR (formerly LCPC), the laboratory of the author, mainly concerns geophysics, civil engineering, material and structural mechanics, thermics, and the reconstruction and improvement of images. These applications will form the primary subject matter of this booklet.

The nature of inverse problems can be better understood by examining their historical evolution. Researchers and engineers working on geophysical data greatly contributed to the development of inverse problem theory. Geophysicists attempt, in effect, to understand the internal behavior of the Earth through data taken at the surface. Before 1970, the developed methods were principally empirical. The work of G. Backus and of F. Gilbert [BG67]-[BG68] near the beginning of the 1970s constitutes the first systematic exploration of the mathematical structure of inverse problems, and is the origin of the development of numerous methods of data interpretation in geophysics. To resolve certain inverse problems concerning the identification of sources described by a linear integral equation of the first kind, these authors proposed a numerical method which now carries their name, and which has given rise to numerous articles and books. This method has often been used for the inversion of seismic data in order to obtain profiles of the density at the interior of the Earth [BG70]. Furthermore, we should equally mention the more recent work of W. Menke [Men89] on the analysis of geophysical data and on discrete inverse theory.

In France, the 1980s witnessed A. Tarantola, one of the precursors of inverse problem theory. Tarantola resolved inverse problems concerning the processing of geophysical data principally by means of probabilistic models (the Bayesian approach). Tarantola is the author of several books [TM82],[Tar87] and of numerous articles [MT02] on this subject. Twelve years later, following the works of H.D. Bui, the use of inverse methods for mechanical and civil engineering became increasingly popular. The research team "Inverse Problems-Identification-Optimization", of the Laboratoire de Mécanique des Solides at l'Ecole Polytechnique, brought together around Bui many active researchers, such as M. Bonnet, A. Constantinescu and C. Stolz (cf. [BC99], [BC05], [BCM04], [PS01], [PS03]). The book [Bui93] is an excellent introduction to inverse problems in material mechanics. The applications of its proposed identification techniques include non-destructive detection; the characterization of internal defects, homogeneities and inhomogeneities; the identification of singularities in fracture mechanics; and the identification of the physical

parameters of materials. A recent special issue of Comptes Rendus from the Académie des Sciences [M0́8], dedicated to H.D. Bui, was devoted to inverse and non-linear problems, in recognition of their importance in present-day solid mechanics.
One can also consult these three Internet sites specifically dedicated to inverse problems. Here one can find useful information such as people working on these problems, announcements of seminars and of past and upcoming conferences, new books, and a selection of relevant scientific articles:

- http://www.me.ua.edu/inverse/, http://www.inverse problems.net/ (Inverse Problems International Association: IPIA),
- http://www.me.ua.edu/inverse/ (University of Alabama)
- http://www.mth.msu.edu/ipnet/ (Inverse Problems Network: IPNet).

Among the scientific reviews dedicated to inverse problems, one can cite:

- Journal of Inverse Problems in Science and Engineering (Taylor Francis Group - G.S. Dulikravich, Editor),
- Journal of Inverse Problems (IOP electronics journals - F. A. Grünbaum. Editor),
- Journal of Inverse and Ill-Posed Problems (Walter de Gruyter - M. M. Lavrentiev. Editor),
- Inverse problems and Imaging (AIMS American Institute of Mathematical Sciences - Lassi Päivärinta. Editor).

One can equally note a recently published special issue of the European Journal of Automated Systems on systems identification.
Finally, in a more modest fashion, it will soon be twenty years since IFSTTAR (previously LCPC)[1] began research concerning the applications of inverse methods in civil and urban engineering. We cite as an example the following theses:

[1]IFSTTAR is the French Institute of Science and Technology for Transport, Development and Networks. It was founded on January 1st, 2011, from a merger of the INRETS Institute and the LCPC Laboratory and has now 1,250 employees. This new entity enjoys the status of a public scientific and technological institution and is overseen by the French Ministry of Ecology, Sustainable Development, Transport and Housing on the one hand and the Ministry of Higher Education and Research on the other. Renowned in the international arena, IFSTTAR conducts applied research and expert appraisals in the fields of transport, infrastructure, natural hazards and urban issues, with the aim of improving the living conditions of French residents and, more broadly, promoting the sustainable development of societies.

- Techniques of seismic tomography with two-dimensional and three-dimensional geometric reconstructions for geophysical reconnaissance: [Cot88],

- Modal identification of structures under vibration, or the estimation of modal parameters (frequencies and fundamental modes of vibration, size of modal damping) of linear mechanical systems with constant parameters. This estimation is found via the system's dynamical responses in the low-frequency range: [Arg90] and [Cré90],

- Identification of the vibratory behavior of buildings from seismic records: [Afr91],

- Identification of parameters in rain forecast models in hydrology: [And98],

- Dynamical monitoring of structures using continuous wavelet analysis. This is performed through the application of the continuous wavelet transform to the transient responses of structures obtained after shock or ambient excitations: [Le03];

as well as a few relatively more recent books about inverse problems, written by researchers at IFSTTAR:

- Non-destructive control techniques, for example for the non-destructive evaluation of the deterioration state of concrete buildings: [BA05],

- The design of experiments: an indispensable tool for experimentation [Lin05],

- In the thesis for authorization of directing [Arg04], one important section concerns the use of the **wavelet transform** for modal identification. Wavelet transform is a fundamental tool for processing measurement signals, rendering much easier the analysis of these signals. With this tool, the process of dynamics parameter identification, and in particular the identification of modal parameters, is greatly simplified.

- A collective publication [APD08] written by researchers at LCPC and their colleagues aims at listing the various techniques used until 2008 by the Ministry of the Environment, Energy, Sustainable Development and Territorial Planning, and in particular those concerning mechanical and civil engineering. Many various inverse problems are considered: identification of mechanical parameters for a compressor, identification of modal parameters or of discrete hysteretic systems, identification of causes, detection and recognition of images, optimization of noise barrier walls, characterization of heterogeneities, and various tomographic problems applied to geophysics and the close examination of artworks. The numerical resolution methods proposed in this book range from the now classic optimization methods for regular or convex cases, to metaheuristic methods of global optimization, for which this book provides a CD with a toolbox in Scilab.

Different Approaches to Solving Inverse Problems

Inversion or inverse methods theory is an ensemble of mathematical techniques that operate on a reduced number of data in a problem, in the hope of obtaining useful information pertinent to the real physical system studied. This information is found using inferences taken from observations. The mathematical techniques used need to take into account the imprecise, redundant and/or partially incomplete character of the data. For this type of problem it is not the mathematical solutions of the term that are searched for, but rather an inventory of the complete ensemble of solutions arising from the near uncertainties. Among the numerous solutions, one makes a choice according to additional criteria (physical plausibility, a priori supplementary information, etc.).
Given a general case, one looks for the stable solution(s), in the sense of Hadamard. According to the basis searched for, the methods of investigating inverse problems can be classified into three main categories:

Functional analysis

The inverse problem, ill posed by its nature, is modified into a well-posed problem by playing with the choice of the spaces that describe the variables, and the choice of their topology that allows the determination of the standard deviation or error. These choices are determined principally by physical, not mathematical, considerations. The corresponding methods generally propose the introduction of global constraints on the classes of solutions.

Regularization of Ill-Posed Problems

The solution obtained by regularized inversion will depend upon the data in a continuous manner, and will approach the exact solution (supposing it exists). The regularization of an inverse problem consists in rewriting the problem. It is based on two guiding principles:

- Define a framework for taking into account a priori supplementary information "exterior to the mathematics."
- Assure that the new solution is stable and will take into account additional information from measurement errors.

To regularize an inverse problem, several methods exist. These can be used singly or in combination. Several examples are:

- Tikhonov's regularization (the most well-known method, cf. [TA77], [HEN96] , which works by adding a non-negative stabilizing functional into the functional to be minimized. The stabilizing functional is able to take the smallest possible positive values and to take into account the available a priori information,
- Reduction of the number of parameters, in order to diminish the sensibility of the criteria to data fluctuations,
- Introduction of constraints (equalities or inequalities confirmed by the unknowns) into the functional to be minimized, in order to obtain only physically acceptable values for the research,
- Filtering of trial data through techniques of signal processing (frequency filters, modified Fourier transforms, time frequency transforms, etc. [CHT98]). Time frequency transforms, such as wavelet analysis (cf. [LA04, AE05, EA07, RmA10]), can be particularly suited to analyze data that change over time,
- The method of quasireversibility, initially proposed by R. Lattès and J.L. Lions [LL93], well-suited for Cauchy problems, such as in elliptical partial differential equations [Bou07]. This method consists in modifying the differential or partial differential operator, generally by changing the order of the derivatives it operates on. This is done in such a way as to obtain a new problem that is well posed for the initial data or for the known boundaries.

Stochastic or Bayesian Inversion

In this case, in order to represent every uncertainty, all of the variables are considered to be random. One is thus interested in the probability density

function. This function is associated with the unknowns and with the data of the problem from which one looks for the characteristic values: average value, value of the largest probability, dispersion, correlations, etc.
As we have mentioned in the preceding paragraph, an inverse problem most often boils down to a problem of minimizing the difference between calculation and experimental results, or to a problem of maximizing likelihood. The inverse problem thus leads to an optimization problem, which is often a large problem in and of itself.
The numerical techniques used to solve an **optimization** problem are as varied as the domains to which inverse problems themselves can be applied (linear or non-linear least squares, maximum plausibility, Monte-Carlo method, linear programming, simulated annealing, genetic or evolutionary algorithms, optimal control methods, etc.). The objective of this introduction is not to give a detailed list of optimization methods. For more information, the reader can consult the abundant literature on this subject, from which we mention the following works: the books of Fletcher [Fle80] and Ciarlet [Cia82] are classics, the work of Culioli [Cul94] is simple and presents the most common numerical methods, that of Bergounious [Ber01] concerns both optimization and control, while [Gro93] is more oriented toward inverse problems. For deterministic methods and convex problems, one might consult [HP95], and for non-deterministic methods and discrete optimization, [PR02]. Finally, in 2008, an encyclopedia in English on optimization was reedited [FP08].

We should note that if a problem causes parameters to occur that assure the regularity and stability of the criteria, it is considered to be within the framework of "classical" optimization. For classical optimization, numerical methods calculate the first and/or second derivatives of the cost function in terms of the unknowns. Iterative algorithms for optimization often in a repetitive fashion call on the numerical solution of an associated direct problem. The direct problem is frequently defined with the help of partial differential equations or systems of differential equations in terms of time. For this, there are an abundant number of solution methods, and to cite but the most common: the finite difference method, the finite element method, and the boundary element method.
In the absence of regularity hypotheses, it becomes difficult to find via classical methods a global optimum or satisfactory local optimums. It is then necessary to use global minimization methods, for example genetic or evolutionary. Effective but not always allowing the use of convergence theorems, these methods constitute what are called zero-order optimization methods or "metaheuristics." Metaheuristics are generally iterative stochastic algorithms, which progress toward the global extreme of the objective function through

sampling. These methods have seen considerable progress these past years due to the development of calculation methods. The recent works [JDT03], in French, and [dC06], in English, present a family of metaheuristics, such as simulated annealing, tabu search, evolutionary and genetic algorithms, and ant colony algorithms, to name only the most important.

Conclusion

Inverse problems form a prolific field. As we have seen, their applications are numerous, the problems diverse and the methods employed inevitably various. Particularly for their formalization and analysis, but also for their method of solution, there exists no “standard”approach to inverse problems. Nevertheless, the formalization and solving of different inverse problems can give rise to ideas, procedures and observations that are useful from one application to the next. To prolong this conclusion, we can state that at this era, when simulation methods achieve startling progress everyday, the steady development of inverse problems invites one to reconsider and rethink acquired classical notions. One immediate example is the concept of measurement, which rests more and more on the interweaving of three elements: experimentation, techniques of direct simulation, and inversion algorithms. The confrontation between model and measurement has thus become a promising pathway toward a better understanding of the studied physical phenomenon, whether it be the surveillance of the evolutionary state of the system, or the validation of imperfectly realized measurements that could interfere with the outcome. To push this idea even further, inverse thinking, by overturning classical notions and encouraging different methods of reasoning, could incite students, engineers and researchers to formulate original ideas and set out on research with daring perspectives.

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Bibliography

[AE05] P. Argoul and S. Erlicher. *Mechanical modelling and computational issues in civil engineering*, volume 23 of *Lecture Notes in Applied and Computational Mechanics*, chapter On the use of continuous wavelet analysis for modal identification, pages 359–368. Springer, 2005.

[Afr91] H. Afra. *Identification du comportement sismique de bâtiments à partir de leurs réponses accélérométriques.* Thèse de doctorat, Ecole Nationale des Ponts et Chaussées, 1991.

[And98] H. Andrieu. *Interprétation de la mesure radar météorologique pour l'hydrologie - Modélisation des bassins versants urbanisés.* Thèse d'habilitation à diriger des recherches, Institut National Polytechnique de Grenoble, 1998. 67 pages + Annexes.

[APD08] P. Argoul, Nelly Point, and G. Dutilleux, editors. *Problèmes inverses en génie civil.* Coll. Etudes et Recherches des LPC, SI 15. LCPC, Paris, 2008. 265 pages + cederom.

[Arg90] P. Argoul. *Identification des structures vibrantes.* Thèse de doctorat, Ecole Nationale des Ponts et Chaussées, 1990. 283 pages.

[Arg04] P. Argoul. *Quelques réflexions sur l'identification de paramètres en dynamique des matériaux et des structures.* Hdr, Claude Bernard University, Lyon 1, 2004.

[BA05] D. Breysse and O. Abraham. *Méthodologie d'évaluation non destructive de l'état d'altération des ouvrages en béton.* Association française de génie civil, 2005.

[BC99] H.D. Bui and A. Constantinescu. Problèmes inverses. de l'expérimentation à la modélisation. ARAGO. OFTA, 1999.

[BC05] M. Bonnet and A. Constantinescu. Inverse problems in elasticity. *Inverse Problems*, 21(2):R1–R50, 2005.

[BCM04] H.D. Bui, A. Constantinescu, and H. Maigre. Numerical identification of linear cracks in 2d elastodynamics using the instantaneous reciprocity gap. *Inverse Problems*, 20(4):993–1001, 2004.

[Ber01] M. Bergounioux. *Optimisation et contrôle optimale des systèmes linéaires*. Dunod, 2001.

[BG67] G.E. Backus and J.F. Gilbert. Numerical applications of a formalism for geophysical inverse problems. *Geophys. J. Roy. Astr. Soc.*, 1967.

[BG68] G.E. Backus and J.F. Gilbert. The resolving power of gross earth data. *Geophys. J. Roy. Astr. Soc.*, 1968.

[BG70] G.E. Backus and J.F. Gilbert. Uniqueness in the inversion of inaccurate gross earth data. *Phil. Trans. Roy. Soc. London Ser. A*, 1970.

[BM91] V. Badeva and V. Morozov. *Problèmes incorrectement posés*. Masson, Paris, 1991.

[Bou07] L. Bourgeois. A stability estimate for ill-posed elliptic cauchy problems in a domain with corners. *C. R. Acad. Sci. Paris, Ser. I*, 345:385–390, 2007.

[Bui93] H.D. Bui. *Introduction aux problèmes inverses en mécanique des matériaux*. Collection de la Direction des Etudes et Recherches d'Electricité de France. Eds Eyrolles, 1993.

[CHT98] R. Carmona, W.-L. Hwang, and B. Torrésani. *Practical time-Frequency analysis*. Academic Press, New York, 1998.

[Cia82] P.G. Ciarlet. *Optimisation et contrôle optimale des systèmes linéaires*. Masson, 1982.

[Cot88] P. Cote. *Tomographies sismiques en génie civil*. Thèse de doctorat d'etat ès sciences physiques, Université de Grenoble, 1988. 193 pages.

[Cré90] C. Crémona. *Recursive time domain modal identification techniques*. Thèse de doctorat, University de Wales, 1990. 304 pages.

[Cul94] J.C. Culioli. *Introduction à l'optimisation*. Ellipses, 1994.

[dC06] L. Nunes de Castro. *Fundamentals of natural computing: basic concepts, algorithms, and applications.* Chapman & Hall/CRC, 1 edition, 6 2006.

[EA07] S. Erlicher and P. Argoul. Modal identification of linear non-proportionally damped systems by wavelet transform. *Mech. Syst. Signal Proc.*, 21(3):1386–1421, 2007.

[Fle80] Fletcher. *Practical methods of optimization.* JohnWiley and Sons, 1980.

[FP08] C.A. Floudas and P.M. Pardalos(Editors). *Encyclopedia of optimization.* Springer, 2nd edition, 2008.

[Gro93] C.W. Groetsch. *Inverse problems in the mathematical sciences.* Vieweg Verlag, Wiesbaden/Braunschweig, Germany, 1993.

[HEN96] M. Hanke H.W. Engl and A. Neubauer. *Regularization of inverse problems.* KAP, 1996.

[HP95] R. Horst and P.M. Pardalos, editors. *Handbook of global optimization.* Kluwer Academic Publishers, 1995.

[JDT03] P. Siarry J. Dréo, A. Pétrowski and E. Taillard. *Métaheuristiques pour l'optimisation difficile.* Algorithmes. Eds Eyrolles, 2003.

[LA04] T.P. Le and P. Argoul. Continuous wavelet transform for modal identification using free decay response. *J. Sound Vib.*, 277(1-2):73–100, 2004.

[Le03] T-P. Le. *Auscultation dynamique des structures à l'aide de l'analyse continue en ondelettes.* PhD thesis, thèse ENPC, E.D. MODES, 2003.

[Lin05] R. Linder. *Les plans d'expériences - Un outil indispensable à l'expérimentation.* 2005.

[LL93] R. Lattès and J.L. Lions. *Méthode de quasi réversibilité et applications.* Eds Eyrolles, 1993.

[M0́8] Comptes-Rendus – Série Mécanique. Duality, inverse problems and nonlinear problems in solid mechanics. volume 336, pages 1–259. janvier 2008.

[Men89] W. Menke. *Geophysical data analysis : Discrete inverse theory, revised version*, volume 45 of *International Geophysics Series.* Academic Press San Diego, 1989.

[MT02] K. Mosegaard and A. Tarantola. Probabilistic approach to inverse problems. chapter International Handbook of Earthquake and Engineering Seismology, pages 237–265. Academic Press, 2002. Part A.

[PR02] P.M. Pardalos and H.E. Romeijn, editors. *Handbook of global optimization - Nonconvex optimization and its applications)*, volume 2. Springer, 1 edition, 6 2002.

[PS01] M. Peigney and C. Stolz. Approche par contrôle optimal du comportement asymptotique des structures élastoviscoplastiques sous chargement cyclique. *C. R. Acad. Sci.*, 329:643–648, 2001.

[PS03] M. Peigney and C. Stolz. An optimal control approach of inelastic structures under cyclic loading. *J. Mech. Phys. Solids*, 51:575–605, 2003.

[RmA10] C. Rouby, D. Rémond, and P. Argoul. Orthogonal polynomials or wavelet analysis for mechanical system direct identification. *Ann. Solid Struct. Mech.*, 1:41Ű58, 2010.

[TA77] A.N. Tikhonov and V.Y. Arsenin. *Solutions to ill-posed problems.* Winston-Wiley, New-York, 1977.

[Tar87] A. Tarantola. *Inverse problem theory.* Elsevier, 1987.

[TM82] A. Tarantola and J. Morlet. Inverse problems=quest for information. *J. Geophys.*, 50:159–170, 1982.